DELTA SETS FOR NUMERICAL SEMIGROUPS WITH EMBEDDING DIMENSION THREE

P. A. GARCÍA-SÁNCHEZ, D. LLENA, AND A. MOSCARIELLO

Abstract. We present a fast algorithm to compute the Delta set of a nonsymmetric numerical semigroups with embedding dimension three.

A monoid is a half-factorial monoid if for every element all the lengths of all the factorizations of this element in terms of atoms remain the same. Delta sets were introducee to measure how far a monoid can be from being half-factorial, and thus how wild the sets of lengths of factorizations are ([11]). Geroldinger in [11] presented the first results on Delta sets, also known as sets of distances, computing in particular the minimum distance between any two factorizations with consecutive lengths. It was shown in [7] that for a monoid with bounded sets of lengths of factorizations, the maximum was reached in a particular class of elements, known as Betti elements (which are important for minimal presentation computations).

Recently Delta sets have been intensively studied on numerical semigroups ([4, 5, 6]). It has been shown that Delta sets are eventually periodic ([4]), and a bound for this periodicity was presented in that paper. As a byproduct, we get a procedure to compute the Delta set of a numerical semigroup (which is the union of all Delta sets of its elements). This bound was improved in [13], then in [10], and lately in [3], where the fastest procedure to compute the Delta set of a numerical monoid is presented, based on dynamic programing. Christopher O'Neil implemented this procedure for the `GAP` ([9]) package `numericalsgps` ([8]). In [6] it is shown that when the generators are too close to each other the Delta set of the numerical semigroup becomes the simplest possible: a singleton.

In the present manuscript we intend to understand better the behavior of Delta sets of element in a nonsymmetric numerical semigroup generated by three elements. As a consequence of this study we answer a question proposed by Scott Chapman during the International Meeting on Numerical Semigroups held in Vila Real on 2012. We are also able to compute the Delta set of these monoids with the same complexity as Euclid's greatest common divisor algorithm. We will show some examples of execution times comparing this new approach with the current implementation in [8] (which is meant for any numerical semigroup).

As it was pointed out in [7], minimal presentations are a fundamental tool to study Delta sets, and we take advantage that minimal presentations of nonsymmetric numerical semigroups with embedding dimension three are well known ([8, Chapter 9]), and are "unique".

1. Preliminaries

Let $\mathbb{N}$ denote the set of nonnegative integers. Given $n_1, n_2, n_3 \in \mathbb{N}$ with $\gcd(n_1, n_2, n_3) = 1$, the *numerical semigroup* generated by $\{n_1, n_2, n_3\}$ is the set $S = \langle n_1, n_2, n_3\rangle = \{x_1n_1 + x_2n_2 + x_3n_3 \mid (x_1, x_2, x_3) \in \mathbb{N}^3\}$, which is a submonoid of $(\mathbb{N}, +)$. We will assume that $n_1 < n_2 < n_3$, and that $\{n_1, n_2, n_3\}$ is a minimal generating system for S, that is, there is no $a, b \in \mathbb{N}$ such that $n_i = an_j + bn_k$

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with $\{i,j,k\}=\{1,2,3\}$. In this setting it is said that S is a semigroup with embedding dimension 3.

The *set of factorizations* of $s\in S$ is $\mathsf{Z}(s)=\{(x_1,x_2,x_3)\in\mathbb{N}^3 \mid x_1n_1+x_2n_2+x_3n_3=s\}$. We denote the *length* of a factorization $\mathbf{x}=(x_1,x_2,x_3)\in\mathsf{Z}(s)$ as $|\mathbf{x}|=x_1+x_2+x_3$. We will use $|\mathbf{x}|=x_1+x_2+x_3$ for any $\mathbf{x}\in\mathbb{Z}^3$. The *set of lengths* of $s\in S$ is $\mathcal{L}(s)=\{|\mathbf{x}| \mid \mathbf{x}\in\mathsf{Z}(s)\}$. It is easy to see that $\mathcal{L}(s)\subset[0,s]$, and consequently $\mathcal{L}(s)$ is finite. So it is of the form $\mathcal{L}(s)=\{m_1,\dots,m_k\}$ for some positive integers $m_1<m_2<\cdots<m_k$. The set

$$\Delta(s)=\{m_i-m_{i-1} \mid 2\le i\le k\}.$$

is known as the *Delta set* of $s\in S$, and the *Delta set* of S is

$$\Delta(S)=\cup_{s\in S}\Delta(s).$$

As a particular instance of [11, Lemma 3], we get the following result.

Theorem 1. *Let S be a numerical semigroup. Then*

$$\min\Delta(M)=\gcd\Delta(M).$$

Set $d=\gcd\Delta(M)$. There exists $k\in\mathbb{N}\setminus\{0\}$ such that

$$\Delta(M)\subseteq\{d,2d,\dots,kd\}.$$

Actually this k is fully determined in our setting in [7].

The goal of this paper is to describe a fast procedure to compute this set. We start recalling some results and definitions.

Given $\{i,j,k\}=\{1,2,3\}$, define

$$c_i=\min\{k\in\mathbb{Z}^+ \mid kn_i\in\langle n_j,n_k\rangle\}.$$

Then there exists $r_{ij},r_{ik}\in\mathbb{N}$ such that

$$c_in_i=r_{ij}n_j+r_{ik}n_k.$$

The condition $\gcd(n_1,n_2,n_3)=1$ is equivalent to $\#(\mathbb{N}\setminus S)<\infty$. Let $\mathrm{F}=\max(\mathbb{Z}\setminus S)$, the *Frobenius number* of S. We say that S is *symmetric* if whenever $x\in\mathbb{Z}\setminus S$, then $\mathrm{F}-x\in S$.

Proposition 2 ([12, Theorem 3])**.** *If S is not symmetric, then the $r_{ij},r_{ik}\in\mathbb{Z}^+$ are unique. Moreover, $c_i=r_{ji}+r_{ki}$.*

From $n_1<n_2<n_3$ we obtain the following result.

Lemma 3. *Under the standing hypothesis, $c_1>r_{12}+r_{13}$ and $c_3<r_{31}+r_{32}$.*

Set

$$\delta_i=|c_i-r_{ij}-r_{ik}|$$

for every $\{i,j,k\}=\{1,2,3\}$. By the previous lemma $\delta_1=c_1-r_{12}-r_{13}$ and $\delta_3=r_{31}+r_{32}-c_3$. Also, from Proposition 2, $\delta_2=|\delta_1-\delta_3|$.

Lemma 4 ([7, Corollary 3.1])**.** *Under the standing hypothesis,* $\min\Delta(S)=\gcd(\delta_1,\delta_3)$ *and* $\max\Delta(S)=\max\{\delta_1,\delta_3\}$.

Remark 5. In light of the last lemma, we can consider $\delta_1\neq\delta_3$ because in other case we will have $\min\Delta(S)=\max\Delta(S)=\delta_1=\delta_3$. And then $\Delta(S)=\{\delta_1\}$.

Define $\varphi\colon\mathbb{N}^3\to S$ as $\varphi(x_1,x_2,x_3)=x_1n_1+x_2n_2+x_3n_3$. Then φ is a monoid epimorphism and thus $S\cong\mathbb{N}^3/\ker\varphi$, where $\ker\varphi=\{(\mathbf{a},\mathbf{b})\in\mathbb{N}^3\times\mathbb{N}^3 \mid \varphi(\mathbf{a})=\varphi(\mathbf{b})\}$. Associated to $\ker\varphi$, we define the subgroup $M=\{\mathbf{a}-\mathbf{b} \mid (\mathbf{a},\mathbf{b})\in\ker\varphi\}$ of $\mathbb{Z}^3$. Notice that if $s\in S$ and $\mathbf{x},\mathbf{y}\in\mathsf{Z}(s)$, then $\mathbf{x}-\mathbf{y}\in M$.

A *presentation* of S is a system of generators of the congruence $\ker\varphi$. It is well known (see for instance [15, Example 8.23]) that

$$\sigma = \{((c_1,0,0),(0,r_{12},r_{13})),((0,c_2,0),(r_{21},0,r_{23})),((0,0,c_3),(r_{31},r_{32},0))\}$$

is a (minimal) presentation of S. It follows easily that if we set

$$\mathbf{v}_1 = (c_1,-r_{12},-r_{13}),\ \mathbf{v}_2 = (-r_{21},c_2,-r_{23}) \text{ and } \mathbf{v}_3 = (r_{31},r_{32},-c_3),$$

then M is generated as a group by $\{\mathbf{v}_1,\mathbf{v}_2,\mathbf{v}_3\}$. In light of Proposition 2, $\mathbf{v}_2 = \mathbf{v}_3 - \mathbf{v}_1$, and consequently we obtain the following result.

Proposition 6. *Let $s \in S$ and $\mathbf{x},\mathbf{y} \in \mathsf{Z}(s)$. Then there exists $\lambda_1,\lambda_3 \in \mathbb{Z}$ such that $\mathbf{x}-\mathbf{y} = \lambda_1\mathbf{v}_1 + \lambda_3\mathbf{v}_3$.*

2. Bézout Couples

A natural way to study $\Delta(S)$ passes through a better understanding of M. This is because $\delta \in \Delta(S)$ if and only if

1. there exists $\mathbf{x},\mathbf{y} \in \mathsf{Z}(s)$ for some $s \in S$, such that $|\mathbf{x}| > |\mathbf{y}|$ and $\delta = |\mathbf{x}| - |\mathbf{y}|$ $(= |\mathbf{x}-\mathbf{y}|)$, and
2. there is no $\mathbf{z} \in \mathsf{Z}(s)$ such that $|\mathbf{x}| > |\mathbf{z}| > |\mathbf{y}|$.

The first condition relies on M and for the second we introduce the concept of Bézout couples.

Proposition 7. *Let $\delta_1,\delta_3 \in \mathbb{Z}^+$ and $g = \gcd(\delta_1,\delta_3)$. Then for every $i \in \mathbb{Z}^+$,*

- *there exists a unique couple $(\lambda_{i1},\lambda_{i3}) \in \mathbb{Z}\times\mathbb{Z}$ such that $\lambda_{i1}\frac{\delta_1}{g} + \lambda_{i3}\frac{\delta_3}{g} = i$ and $0 < \lambda_{i3} \le \frac{\delta_1}{g}$,*
- *there exists a unique couple $(\mu_{i1},\mu_{i3}) \in \mathbb{Z}\times\mathbb{Z}$ such that $\mu_{i1}\frac{\delta_1}{g} + \mu_{i3}\frac{\delta_3}{g} = i$ and $0 < \mu_{i1} \le \frac{\delta_3}{g}$.*

Proof. Follows from elementary number theoretic arguments. $\square$

From now on, we will assume that $\gcd(\delta_1,\delta_3) = 1$, otherwise we normalize δ_1 and δ_3 by $\gcd(\delta_1,\delta_3)$ as in the statement of the last proposition.

Definition 8. *Let $\delta_1,\delta_3 \in \mathbb{Z}^+$ be such that $\gcd(\delta_1,\delta_3) = 1$, and let $i \in \{1,\ldots,\max\{\delta_1,\delta_3\}\}$.*

(1) Define the λ-Bézout couple of $i \in \mathbb{Z}^+$ as the unique couple $(\lambda_{i1},\lambda_{i3}) \in \mathbb{Z}\times\mathbb{Z}$ such that $\lambda_{i1}\delta_1 + \lambda_{i3}\delta_3 = i$ and $0 < \lambda_{i3} \le \delta_1$. We will denote this by $\boldsymbol{\lambda}_i = (\lambda_{i1},\lambda_{i3})$.

(2) Define the μ-Bézout couple of $i \in \mathbb{Z}^+$ as the unique couple $(\mu_{i1},\mu_{i3}) \in \mathbb{Z}\times\mathbb{Z}$ such that $\mu_{i1}\delta_1 + \mu_{i3}\delta_3 = i$ and $0 < \mu_{i1} \le \delta_3$. We will write $\boldsymbol{\mu}_i = (\mu_{i1},\mu_{i3})$.

Set

$$\mathcal{B}^{(\boldsymbol{\lambda})}_{\delta_1,\delta_3} = \{\boldsymbol{\lambda}_i \mid i \in \{1,\ldots,\max\{\delta_1,\delta_3\}\}\} \quad \text{and} \quad \mathcal{B}^{(\boldsymbol{\mu})}_{\delta_1,\delta_3} = \{\boldsymbol{\mu}_i \mid i \in \{1,\ldots,\max\{\delta_1,\delta_3\}\}\}.$$

We will say that a pair is a Bézout couple if it is either a λ-Bézout or a μ-Bézout couple.

We will associate to some particular Bézout couples possible values in the Delta set of S. For this reason, in light of Lemma 4, in the previous Definition we are only interested in the case $i \in \{1,\ldots,\max\{\delta_1,\delta_3\}\}$. Now we give some properties of Bézout couples:

Lemma 9. *If $1 \le i \le \max\{\delta_1,\delta_3\}$,we have*

(1) $-\delta_3 < \lambda_{i1} \le 0$ and $-\delta_1 < \mu_{i3} \le 0$, and

(2) $\lambda_{i1} + \delta_3 = \mu_{i1}$ and $\lambda_{i3} - \delta_1 = \mu_{i3}$.

Proof. (1) As $i \ge 1$, $-\lambda_{i1}\delta_1 < \lambda_{i3}\delta_3 \le \delta_1\delta_3$. So $-\lambda_{i1} < \delta_3$. Similarly we have $-\mu_{i3} < \delta_1$. Since $i \le \max\{\delta_1,\delta_3\}$, we obtain:

- $\lambda_{i1}\delta_1 = i - \lambda_{i3}\delta_3 < i \le \delta_1$, if $\delta_3 < \delta_1$;
- $\lambda_{i1}\delta_1 = i - \lambda_{i3}\delta_3 \le 0$ if $\delta_1 < \delta_3$.

In both cases, we obtain $\lambda_{i1} \le 0$. Similarly $\mu_{i3} \le 0$.

(2) Subtracting both expressions of i from the Definition 8 we obtain:

$$(\lambda_{i1} - \mu_{i1})\delta_1 + (\lambda_{i3} - \mu_{i3})\delta_3 = 0.$$

And, as $\gcd(\delta_1, \delta_3) = 1$ we have that there exists $a \in \mathbb{Z}$ such that $\lambda_{i1} - \mu_{i1} = a\delta_3$ and $\lambda_{i3} - \mu_{i3} = -a\delta_1$. We know that $0 < \lambda_{i3} - \mu_{i3} < \delta_1 + \delta_1 = 2\delta_1$, whence we have that $a = -1$. □

Definition 10. *Let $\boldsymbol{\lambda}_i$ be the λ-Bézout couple of $i \in \{1, \ldots, \max\{\delta_1, \delta_3\}\}$. We say that $\boldsymbol{\lambda}_i$ is irreducible if there is no $j, k \in \{1, \ldots, \max\{\delta_1, \delta_3\}\}$ such that $\boldsymbol{\lambda}_i = \boldsymbol{\lambda}_j + \boldsymbol{\lambda}_k$. Similarly, let $\boldsymbol{\mu}_i$ be the μ-Bézout couple of $i \in \{1, \ldots, \max\{\delta_1, \delta_3\}\}$. We say that $\boldsymbol{\mu}_i$ is irreducible if there is no $j, k \in \{1, \ldots, \max\{\delta_1, \delta_3\}\}$ such that $\boldsymbol{\mu}_i = \boldsymbol{\mu}_j + \boldsymbol{\mu}_k$.*

Remark 11. All λ-Bézout couples of the form $(x, 1)$ and all μ-Bézout couples of the form $(1, y)$ are irreducible Bezout couples.

Lemma 12. *If $\boldsymbol{x}_i = \boldsymbol{x}_j + \boldsymbol{x}_k$, with $\boldsymbol{x} \in \{\boldsymbol{\lambda}, \boldsymbol{\mu}\}$, then $i = j + k$.*

Proof. The proof follows from the definition. □

We will denote

$$\mathcal{I}^{(\boldsymbol{\lambda})}_{\delta_1,\delta_3} = \left\{ \boldsymbol{\lambda}_i \in \mathcal{B}^{(\boldsymbol{\lambda})}_{\delta_1,\delta_3} \;\middle|\; \boldsymbol{\lambda}_i \text{ irreducible} \right\} \quad \text{and} \quad \mathcal{I}^{(\boldsymbol{\mu})}_{\delta_1,\delta_3} = \left\{ \boldsymbol{\mu}_i \in \mathcal{B}^{(\boldsymbol{\mu})}_{\delta_1,\delta_3} \;\middle|\; \boldsymbol{\mu}_i \text{ irreducible} \right\}.$$

Next we translate the concept of irreducibility to the subgroup M.

Given $\mathbf{z} = (z_1, z_2, z_3) \in \mathbb{Z}^3$ we can always write $\mathbf{z} = \mathbf{z}^+ - \mathbf{z}^-$ with $\mathbf{z}^+, \mathbf{z}^- \in \mathbb{N}^3$ and $\mathbf{z}^+ \cdot \mathbf{z}^- = 0$ (dot product).

Lemma 13. *Let $s \in S$ and $\mathbf{x} = (x_1, x_2, x_3) \in \mathsf{Z}(s)$. Let $\boldsymbol{\alpha} = (\alpha_1, \alpha_2, \alpha_3) \in M$. Then $\mathbf{x} + \boldsymbol{\alpha} \in \mathsf{Z}(s)$ if and only if $\mathbf{x} \geq \boldsymbol{\alpha}^-$.*

Proof. Obviously $\mathbf{x} + \boldsymbol{\alpha} \in \mathsf{Z}(s)$ if and only if $x_j + \alpha_j \geq 0$ for $j \in \{1, 2, 3\}$, and this happens if and only if $x_j \geq -\alpha_j$ whenever $\alpha_j \leq 0$. This is equivalent to $x_j \geq \alpha_j^-$. □

Definition 14. *Let $\boldsymbol{x}_i = (x_{i1}, x_{i3}) \subseteq \mathbb{Z}^2$ be such that $x_{i1}\delta_1 + x_{i3}\delta_3 = i$, for some $i \in \{1, \ldots, \max\{\delta_1, \delta_3\}\}$. Denote by $\tau_{\boldsymbol{x}_i}$ the vector*

$$\tau_{\boldsymbol{x}_i} = (\tau_{i1}, \tau_{i2}, \tau_{i3}) := x_{i1}\mathbf{v}_1 + x_{i3}\mathbf{v}_3 \in M.$$

Remark 15. Observe that $|\tau_{\boldsymbol{x}_i}| = x_{i1}|\mathbf{v}_1| + x_{13}|\mathbf{v}_3| = x_{i1}\delta_1 + x_{13}\delta_3 = i$.

Lemma 16. *Let $\boldsymbol{x}_i = (x_{i1}, x_{i3}) \in \mathbb{Z}^2$ be such that $x_{i1}\delta_1 + x_{i3}\delta_3 = i$, with $i \in \{1, \ldots, \max\{\delta_1, \delta_3\}\}$. Then $x_{i1} \leq 0$ if and only if $\tau_{i2} > 0$.*

Proof. Since $x_{i1}\delta_1 + x_{i3}\delta_3 = i$, the condition $x_{i1} \leq 0$ forces $x_{i3} > 0$. So $-x_{i1}r_{12} + x_{i3}r_{32} > 0$. Analogously, $x_{i1} > 0$ implies $x_{i3} \leq 0$, whence $-x_{i1}r_{12} + x_{i3}r_{32} < 0$. □

Lemma 17. *Let $\boldsymbol{x}_i = (x_{i1}, x_{i3}) \in \mathbb{Z}^2$ be such that $x_{i1}\delta_1 + x_{i3}\delta_3 = i$, with $i \in \{1, \ldots, \max\{\delta_1, \delta_3\}\}$. If $\delta_j < \delta_k$, with $\{j, k\} = \{1, 3\}$, then $|x_{ik}| \leq |x_{ij}|$.*

Proof. We have that $0 < x_{ij}\delta_j + x_{ik}\delta_k = i \leq \delta_k$. We can divide by δ_k to obtain: $0 < x_{ij}\frac{\delta_j}{\delta_k} + x_{ik} \leq 1$. If $x_{ik} \leq 0$, we have $0 \leq |x_{ik}| = -x_{ik} < x_{ij}\frac{\delta_j}{\delta_k} < x_{ij} = |x_{ij}|$. While if $x_{ik} > 0$, then $x_{ij} < \frac{\delta_j}{\delta_k}x_{ij} \leq 1 - x_{ik} \leq 0$. So $x_{ij} \leq -x_{ik} = -|x_{ik}|$. □

Corollary 18. *Let $\boldsymbol{x}_i = (x_{i1}, x_{i3}) \in \mathbb{Z}^2$ be such that $x_{i1}\delta_1 + x_{i3}\delta_3 = i$, with $i \in \{1, \ldots, \max\{\delta_1, \delta_3\}\}$. The following table describes the signs of both x_{i3} and the coordinates of $\tau_{\boldsymbol{x}_i}$.*

	$x_{i1} \leq 0$	$x_{i1} > 0$
$\delta_1 > \delta_3$	$\tau_{i2} > 0, \tau_{i3} < 0, x_{i3} > 0$	$\tau_{i2} < 0, \tau_{i3} > 0, x_{i3} \leq 0$
$\delta_3 > \delta_1$	$\tau_{i2} > 0, \tau_{i1} < 0, x_{i3} > 0$	$\tau_{i2} < 0, \tau_{i1} > 0, x_{i3} \leq 0$

Proof. If $x_{i1} \leq 0$ we have $x_{i3} > 0$.

$\delta_1 > \delta_3$**:** from Lemma 17, we have $0 \le -x_{i1} \le x_{i3}$; as $c_3 > r_{31}$, we obtain $\tau_{i3} = -x_{i1}r_{13} - x_{i3}c_3 < 0$.

$\delta_1 < \delta_3$**:** in view of Lemma 17 again, we have that $0 \le x_{i3} \le -x_{i1}$; from $c_1 > r_{13}$, we obtain $\tau_{i1} < 0$.

Lemma 16 states that $\tau_{i2} > 0$.

The case $x_{i1} > 0$ is analogous. □

Lemma 19. *Let $\boldsymbol{x}_i$ be a reducible Bézout couple, with $\boldsymbol{x} \in \{\boldsymbol{\lambda}, \boldsymbol{\mu}\}$ and $1 \le i \le \max\{\delta_1, \delta_3\}$. Then exists $l \in \mathbb{Z}^+$, $l < i$, such that $\tau^-_{\boldsymbol{x}_l} \le \tau^-_{\boldsymbol{x}_i}$.*

Proof. Assume that $\boldsymbol{x}_i = \boldsymbol{\lambda}_i$ (the other case is analogous). As $\boldsymbol{\lambda}_i$ is reducible, it follows that there exist positive integers j and k such that $\boldsymbol{\lambda}_i = \boldsymbol{\lambda}_j + \boldsymbol{\lambda}_k$. Lemma 12 ensures that $k, j < i$, and it is easy to derive that

$$\tau_{\boldsymbol{\lambda}_i} = \tau_{\boldsymbol{\lambda}_j} + \tau_{\boldsymbol{\lambda}_k}. \tag{1}$$

From Lemma 16, $\tau_{i2} > 0$, $\tau_{j2} > 0$ and $\tau_{k2} > 0$, so $\tau^-_{i2} = \tau^-_{j2} = \tau^-_{k2} = 0$. For the other coordinates, we distinguish two cases.

(i) $\delta_1 > \delta_3$. From Corollary 18 we have $\tau_{i3} < 0$, $\tau_{j3} < 0$ and $\tau_{k3} < 0$. Observe that $\lambda_{i3} \ne 0$ because otherwise $|\lambda_{i1}| \le |\lambda_{i3}| = 0$, and thus $i = 0$. So, as $\tau_{i3} = \tau_{j3} + \tau_{k3}$, and they are all negative, we deduce $\tau^-_{i3} = \tau^-_{j3} + \tau^-_{k3}$. Hence $\tau^-_{i3} \ge \tau^-_{j3}$ and $\tau^-_{i3} \ge \tau^-_{k3}$.

From the equation (1) we have that $\tau_{i1} = \tau_{j1} + \tau_{k1}$. Thus, if $\tau_{i1}, \tau_{j1}, \tau_{k1}$ have all the same sign we can deduce, as before, $\tau^-_{j1} \le \tau^-_{i1}$ and $\tau^-_{k1} \le \tau^-_{i1}$, and in this case we can take $l = j$ or $l = k$ to finish the proof. While in the other case one and only one between τ_{j1}, τ_{k1} should be nonnegative (in other cases both nonpositive or both nonnegative implies that $\tau_{i1}, \tau_{j1}, \tau_{k1}$ have the same sign). We call it τ_{l1}. So, for this l we have $\tau^-_{l1} = 0 \le \tau^-_{i1}$, $0 = \tau^-_{l2} \le \tau^-_{i2} = 0$ and $\tau^-_{l3} \le \tau^-_{i3}$.

(ii) $\delta_1 < \delta_3$. Again from Corollary 18, $\tau_{i1} < 0$, $\tau_{j1} < 0$ and $\tau_{k1} < 0$. Hence from $\tau_{i1} = \tau_{j1} + \tau_{k1}$ we deduce $\tau^-_{i1} = \tau^-_{j1} + \tau^-_{k1}$. This leads to $\tau^-_{i1} \ge \tau^-_{j1}$ and $\tau^-_{i1} \ge \tau^-_{k1}$.

Now arguing as above, but with $\tau_{i3} = \tau_{j3} + \tau_{k3}$, we have again two possibilities.

(a) The integers $\tau_{i3}, \tau_{j3}, \tau_{k3}$ have the same sign. We can take $l = j$ or $l = k$ to finish the proof.

(b) One of the τ_{j3}, τ_{k3} must be nonnegative. We choose it to conclude the proof. □

Lemma 20. *With the same notation as the above lemma we have that exists $l \in \mathbb{Z}^+$, $l < i$, such that $\tau^+_{\boldsymbol{x}_l} \le \tau^+_{\boldsymbol{x}_i}$.*

Proof. Analogous to the preceding lemma. □

We denote

$$I_{\delta_1,\delta_3} = I^{(\boldsymbol{\lambda})}_{\delta_1,\delta_3} \cup I^{(\boldsymbol{\mu})}_{\delta_1,\delta_3}.$$

Theorem 21. *Let S be a nonsymmetric numerical semigroup minimally generated by $\{n_1, n_2, n_3\}$ with $n_1 < n_2 < n_3$. Let δ_1, δ_2 and I_{δ_1,δ_2} be defined as above. Let $g = \gcd(\delta_1, \delta_3)$. Then*

$$\Delta(S) = \left\{ gi \in \mathbb{N} \mid i \in \{1, \ldots, \max\{\delta_1/g, \delta_3/g\}, \boldsymbol{x}_i \in I_{\delta_1/g, \delta_3/g} \right\}.$$

Proof. For sake of simplicity assume that $g = 1$.

$\subseteq$. Let $m \in \Delta(S)$. It is obvious that $1 \le m \le \max\{\delta_1, \delta_3\}$.

Assume that $\boldsymbol{\lambda}_m \notin I^{(\boldsymbol{\lambda})}_{\delta_1,\delta_3}$ and $\boldsymbol{\mu}_m \notin I^{(\boldsymbol{\mu})}_{\delta_1,\delta_3}$. Since $m \in \Delta(S)$ there exists $s \in S$, $\mathbf{x} = (x_1, x_2, x_3), \mathbf{y} = (y_1, y_2, y_3) \in \mathsf{Z}(s)$ such that $|\mathbf{y}| - |\mathbf{x}| = m$ and there is no $\mathbf{z} \in \mathsf{Z}(s)$ such that $|\mathbf{x}| < |\mathbf{z}| < |\mathbf{y}|$. By Proposition 6 we have

$$\mathbf{y} - \mathbf{x} = a_1(c_1, -r_{12}, -r_{13}) + a_3(r_{31}, r_{32}, -c_3) = a_1\mathbf{v}_1 + a_3\mathbf{v}_3$$

for some $a_1, a_3 \in \mathbb{Z}$. By taking lengths, we have $a_1\delta_1 + a_3\delta_3 = m$. As $1 \le m \le \max\{\delta_1, \delta_3\}$, we deduce that $a_1 a_3 < 0$. Also, $(a_1 + q\delta_3)\delta_1 + (a_3 - q\delta_1)\delta_3 = m$ for all $q \in \mathbb{Z}$.

For sake of simplicity, write $\mathbf{v} = -\delta_3\mathbf{v}_1 + \delta_1\mathbf{v}_3 = (v_1, v_2, v_3)$. As in Lemmas 16 and 17, we can deduce:

- $v_2 = \delta_3 r_{12} + \delta_1 r_{32} > 0$.
- $v_1 = -\delta_3 c_1 + \delta_1 r_{31}$; thus if $\delta_3 > \delta_1$, then $v_1 < 0$.
- $v_3 = \delta_3 r_{13} - \delta_1 c_3$; whence if $\delta_3 < \delta_1$, then $v_3 < 0$.

(a) $a_1 \le 0$, then $a_3 > 0$ and notice that we can take $q \in \mathbb{N}$ such that $(a_1 + q\delta_3, a_3 - q\delta_1)$ is a λ-Bézout couple. Write

$$\mathbf{y} - \mathbf{x} = (a_1 + q\delta_3)\mathbf{v}_1 + (a_3 - q\delta_1)\mathbf{v}_3 - q\delta_3\mathbf{v}_1 + q\delta_1\mathbf{v}_3 = \tau_{\boldsymbol{\lambda}_m} + q\mathbf{v}.$$

We going to prove that either $\mathbf{y} - \tau_{\boldsymbol{\lambda}_m}$ or $\mathbf{x} + \tau_{\boldsymbol{\lambda}_m}$ are in $\mathsf{Z}(s)$. From Lemma 13, it suffices to show that either $\mathbf{y} > \tau^+_{\boldsymbol{\lambda}_m}$ or $\mathbf{x} > \tau^-_{\boldsymbol{\lambda}_m}$ (observe that $(-\tau_{\boldsymbol{\lambda}_m})^- = \tau^+_{\boldsymbol{\lambda}_m}$). We have, from Corollary 18 and the above remark on (v_1, v_2, v_3) that $\tau_{m2} > 0$, $y_2 - x_2 > 0$ and $v_2 > 0$. Also $y_2 - x_2 = qv_2 + \tau_{m2}$ with $q \ge 0$. So we can deduce $y_2 > qv_2 + \tau_{m2} \ge \tau_{m2} > 0$. Now depending on $\delta_1 < \delta_3$ or $\delta_3 < \delta_1$, we can assure, again from Corollary 18 and the above remark that for $i = 1$ or $i = 3$ we have $\tau_{mi} < 0$, $y_i - x_i < 0$ and $v_i < 0$. Hence $-x_i < y_i - x_i = qv_i + \tau_{mi} < \tau_{mi} < 0$.

We have in this case ($a_1 \le 0$) that $0 < \tau_{m2} < y_2$ and $x_i < \tau_{mi} < 0$. Take j such that $\{i, j\} = \{1, 3\}$. Now, if $\tau_{mj} \le 0$, we have $\tau^+_{\boldsymbol{\lambda}_m} = (0, \tau_{m2}, 0) < \mathbf{y}$, and if $\tau_{mj} > 0$, then $\tau^-_{\boldsymbol{\lambda}_m} = -\tau_{mi}\boldsymbol{e}_i < (x_1, x_2, x_3) = \mathbf{x}$; where $\boldsymbol{e}_i$ is the ith row of the 3×3 identity matrix.

- If $\mathbf{y} > \tau^+_{\boldsymbol{\lambda}_m}$, by Lemma 13 we have $\mathbf{y} - \tau_{\boldsymbol{\lambda}_m} \in \mathsf{Z}(s)$. As $\boldsymbol{\lambda}_m$ is reducible, by Lemma 20, there exists $j \in \mathbb{Z}^+, j < m$ such that $\tau^+_{\boldsymbol{\lambda}_m} \ge \tau^+_{\boldsymbol{\lambda}_j} = (-\tau_{\boldsymbol{\lambda}_j})^-$. As $\mathbf{y} > \tau^+_{\boldsymbol{\lambda}_m}$, in light of Lemma 13, we have that $\mathbf{z} = \mathbf{y} - \tau_{\boldsymbol{\lambda}_j} \in \mathsf{Z}(s)$.
- If $\tau^-_{\boldsymbol{\lambda}_m} < \mathbf{x}$, again by Lemma 13 we deduce $\mathbf{x} + \tau_{\boldsymbol{\lambda}_j} \in \mathsf{Z}(s)$. By Lemmas 19 and 13, we derive $\mathbf{z} = \mathbf{x} + \tau_{\boldsymbol{\lambda}_j} \in \mathsf{Z}(s)$.

In both cases $|\mathbf{x}| < |\mathbf{z}| < |\mathbf{y}|$, which is a contradiction.

(b) $a_1 > 0$. This case is identical, considering now q a nonpositive integer such that $(a_1+q\delta_3, a_3-q\delta_1)$ is a μ-Bézout couple.

$\supseteq$. Let $m \in \{i \in \mathbb{N} \mid 1 \le i \le \max\{\delta_1, \delta_3\}, \boldsymbol{x}_i \in I_{\delta_1,\delta_3}\}$. Thus we have $\boldsymbol{x}_m \in I_{\delta_1,\delta_3}$. Assume to the contrary that $m \notin \Delta(S)$.

We know that $\tau^-_{\boldsymbol{x}_m}$ and $\tau^+_{\boldsymbol{x}_m}$ are factorizations for some $s \in S$ and, from Remark 15, we have $|\tau_{\boldsymbol{x}_m}| = |\tau^+_{\boldsymbol{x}_m}| - |\tau^-_{\boldsymbol{x}_m}| = m$. Hence $|\tau^+_{\boldsymbol{x}_m}| = |\tau^-_{\boldsymbol{x}_m}| + m > |\tau^-_{\boldsymbol{x}_m}|$. Since $m \notin \Delta(S)$, there exists some $\mathbf{z} \in \mathsf{Z}(s)$ such that $|\tau^-_{\boldsymbol{x}_m}| < |\mathbf{z}| < |\tau^+_{\boldsymbol{x}_m}|$. By Proposition 6, we know that there exists $(a_1, a_3), (b_1, b_3) \in \mathbb{Z}^2$ such that

$$\mathbf{z} - \tau^-_{\boldsymbol{x}_m} = a_1\mathbf{v}_1 + a_3\mathbf{v}_3, \quad \tau^+_{\boldsymbol{x}_m} - \mathbf{z} = b_1\mathbf{v}_1 + b_3\mathbf{v}_3,$$

and consequently

$$0 < a_1\delta_1 + a_3\delta_3 < m \text{ and } 0 < b_1\delta_1 + b_3\delta_3 < m.$$

Observe that $\mathbf{z} - \tau^-_{\boldsymbol{x}_m}, \tau^+_{\boldsymbol{x}_m} - \mathbf{z} \notin \mathbb{N}^3$, since any two factorizations of the same element are incomparable.

Notice that, since $\{\boldsymbol{v}_1, \boldsymbol{v}_3\}$ is a basis of M,

$$\boldsymbol{x}_m = (x_{m1}, x_{m3}) = (a_1, a_3) + (b_1, b_3). \tag{2}$$

We will prove in Lemma 23 that both (a_1, a_3) and (b_1, b_3) are Bézout couples. Moreover, this lemma states that if $\boldsymbol{x}_m$ is a λ-Bézout couple, then (a_1, a_3) is a λ-Bézout couple and (b_1, b_3) is a Bézout couple. And, if $\boldsymbol{x}_m$ is a μ-Bézout couple, then (b_1, b_3) is a μ-Bézout couple and (a_1, a_3) is a Bézout couple.

Assume, then, that $\boldsymbol{x}_m = \boldsymbol{\lambda}_m$. It follows from the above mentioned Lemma 23, that (a_1, a_3) is a λ-Bézout couple and (b_1, b_3) is a Bézout couple. It is clear that (b_1, b_3) can not be a λ-Bézout couple

because $\boldsymbol{x}_m = \boldsymbol{\lambda}_m$ is irreducible. Hence in this setting (b_1, b_3) is a μ-Bézout couple. As $\boldsymbol{x}_m = \boldsymbol{\lambda}_m$, from Lemma 16 we have $\tau_{m2} > 0$, so if we write $\mathbf{z} = (z_1, z_2, z_3)$, we have

$$\begin{aligned}\tau_{\boldsymbol{\lambda}_m} &= (\tau_{m1}, \tau_{m2}, \tau_{m3}),\\ \mathbf{z} - \tau^-_{\boldsymbol{\lambda}_m} &= (z_1 - \tau^-_{m1}, z_2, z_3 - \tau^-_{m3}),\\ \tau^+_{\boldsymbol{\lambda}_m} - \mathbf{z} &= (\tau^+_{m1} - z_1, \tau_{m2} - z_2, \tau^+_{m3} - z_3).\end{aligned}$$

It follows that

$$(\tau_{m1}, \tau_{m2}, \tau_{m3}) = (z_1 - \tau^-_{m1}, z_2, z_3 - \tau^-_{m3}) + (\tau^+_{m1} - z_1, \tau_{m2} - z_2, \tau^+_{m3} - z_3).$$

If we apply Corollary 18 to (a_1, a_3) and (b_1, b_3), we deduce the following.

(1) If $\delta_1 < \delta_3$, then $\tau_{m1} < 0$ and $\tau^+_{m1} - z_1 > 0$, so we have $z_1 < \tau^+_{m1} = 0$, contradicting that $\mathbf{z} \in \mathsf{Z}(s) \subseteq \mathbb{N}^3$.

(2) If $\delta_3 < \delta_1$, then $\tau_{m3} < 0$ and $\tau^+_{m3} - z_3 > 0$, so we have $z_3 < \tau^+_{m3} = 0$, which yields again a contradiction.

The case $\boldsymbol{x}_m = \boldsymbol{\mu}_m$ is analogous. □

In order to make the proof of Theorem 21, we have extracted Lemma 23 from it. We need an extra lemma to prove this piece.

Lemma 22. *Let $m \leq \max\{\delta_1, \delta_3\}$. Let (x_{m1}, x_{m3}) be a Bézout couple such that $(x_{m1}, x_{m3}) = (a_1, a_3) + (b_1, b_3)$ with $0 < a_1\delta_1 + a_3\delta_3 < m$ and $0 < b_1\delta_1 + b_3\delta_3 < m$.*

(1) If $a_3 \leq -\delta_1$, then $a_1 > \delta_3$. Moreover, if $\delta_3 > \delta_1$, the converse is also true.
(2) If $a_1 \leq -\delta_3$, then $a_3 > \delta_1$. Moreover, if $\delta_1 > \delta_3$, the converse holds.
(3) If $\boldsymbol{x}_m = \boldsymbol{\lambda}_m$, we have that $a_3 \leq -\delta_1$ implies $\delta_1 < b_3$; while $\delta_1 < b_3$ implies $a_3 < 0$.
(4) If $\boldsymbol{x}_m = \boldsymbol{\lambda}_m$, then the inequality $a_1 > \delta_3$ implies $b_1 < -\delta_3$; while $b_1 \leq -\delta_3$ implies $0 < a_1$.
(5) If $\boldsymbol{x}_m = \boldsymbol{\mu}_m$, then $a_1 \leq -\delta_3$ implies $\delta_3 < b_1$; while $\delta_3 < b_1$ implies $a_1 < 0$.
(6) If $\boldsymbol{x}_m = \boldsymbol{\mu}_m$, we have that $a_3 > \delta_1$ implies $b_3 < -\delta_1$; and $b_3 \leq -\delta_1$ implies $0 < a_3$.

Clearly, the above statements are true if we swap a_i and b_i.

Proof. (1) As $0 < a_1\delta_1 + a_3\delta_3$, if $a_3 \leq -\delta_1$, then $0 < a_1\delta_1 + a_3\delta_3 \leq a_1\delta_1 - \delta_1\delta_3 = (a_1 - \delta_3)\delta_1$. Hence $0 < (a_1 - \delta_3)\delta_1$ and as $\delta_1 > 0$, we deduce that $a_1 > \delta_3$.

Now assume that $\delta_3 > \delta_1$. Since $a_1\delta_1 + a_3\delta_3 < m \leq \max\{\delta_1, \delta_3\}$, if $a_1 > \delta_3$, then $m \geq a_1\delta_1 + a_3\delta_3 > \delta_3\delta_1 + a_3\delta_3 = (\delta_1 + a_3)\delta_3$. We have $\delta_3 \geq m > (\delta_1 + a_3)\delta_3$, whence $1 > \delta_1 + a_3$, or equivalently, $0 \geq \delta_1 + a_3$ and so $a_3 \leq -\delta_1$.

(2) This case is analogous.

(3) Remember that $\boldsymbol{x}_m = \boldsymbol{\lambda}_m$ implies $0 < x_{m3} = a_3 + b_3 \leq \delta_1$. So, if $a_3 \leq -\delta_1$, we have $0 < a_3 + b_3 \leq -\delta_1 + b_3$, and then $\delta_1 < b_3$. If $\delta_1 < b_3$, we obtain $a_3 + \delta_1 < a_3 + b_3 \leq \delta_1$, and then $a_3 < 0$.

(4) If $\boldsymbol{x}_m = \boldsymbol{\lambda}_m$, we have too that $-\delta_3 < x_{m1} = a_1 + b_1 \leq 0$ (Lemma 9). If $a_1 > \delta_3$, then $0 \geq a_1 + b_1 > \delta_3 + b_1$, and so $b_1 < -\delta_3$; while if $b_1 \leq -\delta_3$, we have $-\delta_3 < a_1 + b_1 \leq a_1 - \delta_3$, and then $0 < a_1$.

(5) This case is similar as case (3)

(6) The proof is analogous to case (4). □

Now, we are ready to proof the necessary result to finish the Theorem 21.

Lemma 23. *Consider, as in the proof of Theorem 21 that*

$$\mathbf{z} - \tau^-_{\boldsymbol{x}_m} = a_1\mathbf{v}_1 + a_3\mathbf{v}_3 \text{ and } \tau^+_{\boldsymbol{x}_m} - \mathbf{z} = b_1\mathbf{v}_1 + b_3\mathbf{v}_3,$$

with $\mathbf{z} = (z_1, z_2, z_3) \in \mathbb{N}^3$.

(1) If (x_{m1}, x_{m3}) and (a_1, a_3) are both λ-Bézout couples, then (b_1, b_3) is a Bézout couple. Similarly, if (x_{m1}, x_{m3}) and (b_1, b_3) are both μ-Bézout couples, then (a_1, a_3) is a Bézout couple.

(2) If $\boldsymbol{x}_m = \boldsymbol{\lambda}_m$, then (a_1, a_3) is a λ-Bézout couple, and if $\boldsymbol{x}_m = \boldsymbol{\mu}_m$, then (b_1, b_3) is a μ-Bézout couple.

Proof. (1) If both are λ-Bézout couples, we have by definition and Lemma 9: $-\delta_3 < x_{m1} \leq 0$, $0 < x_{m3} \leq \delta_1$, $0 \leq -a_1 < \delta_3$, and $-\delta_1 \leq -a_3 < 0$. Hence $-\delta_3 < x_{m1} - a_1 = b_1 < \delta_3$ and $-\delta_1 < x_{m3} - a_3 = b_3 < \delta_1$.

As $0 < m = b_1\delta_1 + b_3\delta_3 \leq \max\{\delta_1, \delta_3\}$, we deduce $b_1 b_3 \leq 0$. This proves that (b_1, b_3) is a Bézout couple.

If both are μ-Bézout couples, the proof is similar.

(2) Recall that by Lemma 16 that, $\boldsymbol{x}_m = \boldsymbol{\lambda}_m$ if and only if $\tau_{m2} > 0$. Hence $\mathbf{z} - \tau^-_{\boldsymbol{x}_m} = (z_1 - \tau^-_{m1}, z_2, z_3 - \tau^+_{m3})$. So we have $-a_1 r_{12} + a_3 r_{32} = z_2 > 0$. We distinguish two cases.

- $\delta_1 > \delta_3$. We prove that $-\delta_1 < a_3 \leq \delta_1$. Suppose to the contrary that
  - (i) $a_3 \leq -\delta_1$, from Lemma 22 (1) we have $a_1 > \delta_3$ and so $z_2 < 0$, which is a contradiction; or
  - (ii) $a_3 > \delta_1$, from Lemma 22
    - (2) (sufficient condition) implies $a_1 \leq -\delta_3$,
    - (3) (swapping a and b) yields $b_3 < 0$,
    - (4) (swapping a and b) forces $0 < b_1$.

    As $\delta_1 > \delta_3$, from Corollary 18, $\tau_{m3} < 0$. Then $\tau^+_{\boldsymbol{x}_m} - \mathbf{z} = (\tau^+_{m1} - z_1, \tau^+_{m2} - z_2, -z_3)$. So we have $-b_1 r_{13} - b_3 c_3 = -z_3 < 0$, and, as we are assuming $\delta_1 > \delta_3$, we have $b_1\delta_1 + b_3\delta_3 < m \leq \delta_1$. This implies $(b_1 - 1)\delta_1 < -b_3\delta_3 < -b_3\delta_1$ and then $b_1 - 1 < -b_3$, or equivalently, $b_1 \leq -b_3$. Since $c_3 > r_{13}$, it follows that $b_1 r_{13} + b_3 c_3 < 0$ obtaining again a contradiction.

  The case $\delta_3 > \delta_1$ is analogous.

So, in this case (a_1, a_3) is a Bézout couple. And as, $z_2 > 0$ for Lemma 16 (a_1, a_3) is a λ-Bézout couple. And from Lemma 23 (1), we assure that (b_1, b_3) is too a Bézout couple.

The case $\boldsymbol{x}_m = \boldsymbol{\mu}_m$ is completely similar as $\boldsymbol{x}_m = \boldsymbol{\lambda}_m$ case. □

Example 24. Take $S = \langle 8, 41, 79\rangle$. We have $\delta_1 = 15 - 1 - 1 = 13$ and $\delta_3 = 4 + 5 - 3 = 6$. Following Theorem 21, we can compute a *Bézout Table* for this couple.

Irr.	λ_{i1}	λ_{i3}	i	μ_{i1}	μ_{i3}	Irr.
✓	-5	11	1	1	-2	✓
✓	-4	9	2	2	-4	×
✓	-3	7	3	3	-6	×
✓	-2	5	4	4	-8	×
✓	-1	3	5	5	-10	×
✓	0	1	6	6	-12	×
×	-5	12	7	1	-1	✓
×	-4	10	8	2	-3	×
×	-3	8	9	3	-5	×
×	-2	6	10	4	-7	×
×	-1	4	11	5	-9	×
×	0	2	12	6	-11	×
×	-5	13	13	1	0	✓

TABLE 1. Bézout Table for $\delta_1 = 13$ and $\delta_3 = 6$,

Thus, $\Delta(S) = \{1, 2, 3, 4, 5, 6, 7, 13\}$.

Let $d = \max\{\delta_1, \delta_3\}$. Notice that this procedure has at least $d\log(d)$ complexity, and requires the precomputation of δ_1 and δ_3 . We will try to improve this in the next section. However, we can

get some interesting theoretical consequences out of this (which was the initial motivation to write this manuscript). By using Theorem 21 we can prove two conjectures proposed by Malyshev [13]. Some partial solutions were provided in [2].

Corollary 25. *Let S be a nonsymmetric numerical semigroup with embedding dimension three and $|\Delta(S)| > 1$. If $1 = \min \Delta(S)$, then $\{2, 3\} \subseteq \Delta(S)$.*

Proof. Suppose that $2 \notin \Delta(S)$. By Theorem 21, $\boldsymbol{\lambda}_2, \boldsymbol{\mu}_2 \notin I_{\delta_1,\delta_3}$. Hence (recall that $\boldsymbol{\lambda}_i = \boldsymbol{\lambda}_j + \boldsymbol{\lambda}_k$ implies $i = j + k$). We must then have $\boldsymbol{\lambda}_2 = 2\boldsymbol{\lambda}_1$ and $\boldsymbol{\mu}_2 = 2\boldsymbol{\mu}_1$. However, since $\boldsymbol{\lambda}_l + (\delta_3, -\delta_1) = \boldsymbol{\mu}_l$ for every l (Lemma 9), this implies

$$\boldsymbol{\lambda}_1 + \boldsymbol{\mu}_1 = 2\boldsymbol{\lambda}_1 + (\delta_3, -\delta_1) = \boldsymbol{\lambda}_2 + (\delta_3, -\delta_1) = \boldsymbol{\mu}_2 = 2\boldsymbol{\mu}_1.$$

It follows that $\boldsymbol{\mu}_1 = \boldsymbol{\lambda}_1$, contradicting the definition.

Now assume that $3 \notin \Delta(S)$. By Theorem 21 $\boldsymbol{\lambda}_3, \boldsymbol{\mu}_3 \notin I_{\delta_1,\delta_3}$. Hence we must have $\boldsymbol{\lambda}_3 = \boldsymbol{\lambda}_1 + \boldsymbol{\lambda}_2$ and $\boldsymbol{\mu}_3 = \boldsymbol{\mu}_1 + \boldsymbol{\mu}_2$. Here we obtain

$$\boldsymbol{\lambda}_1 + \boldsymbol{\mu}_2 = \boldsymbol{\lambda}_1 + \boldsymbol{\lambda}_2 + (\delta_3, -\delta_1) = \boldsymbol{\lambda}_3 + (\delta_3, -\delta_1) = \boldsymbol{\mu}_3 = \boldsymbol{\mu}_1 + \boldsymbol{\mu}_2,$$

and thus $\boldsymbol{\mu}_1 = \boldsymbol{\lambda}_1$, yielding again a contradiction. □

Remark 26. In the proof of the above corollary the contradiction is reached once we obtain that $\boldsymbol{\lambda}_i = \boldsymbol{\lambda}_j + \boldsymbol{\lambda}_k$ and $\boldsymbol{\mu}_i = \boldsymbol{\mu}_j + \boldsymbol{\mu}_k$ for some $i, j, k \in \mathbb{Z}^+$.

So we cannot guarantee that $4 \in \Delta(S)$ under the same assumptions, because in counterexamples such as $S = \langle 7, 18, 19\rangle$ we get $\boldsymbol{\mu}_4 = \boldsymbol{\mu}_1 + \boldsymbol{\mu}_3$ and $\boldsymbol{\lambda}_4 = \boldsymbol{\lambda}_2 + \boldsymbol{\lambda}_2$, whence $\boldsymbol{\lambda}_4 \notin I_{\delta_1,\delta_3}$ and $\boldsymbol{\mu}_4 \notin I_{\delta_1,\delta_3}$. So $4 \notin \Delta(S)$.

Under some extra assumptions we can get more information.

Corollary 27. *If, in addition, $\min\{\delta_1, \delta_3\} = 1$, then $\Delta(S) = \{1, 2, \ldots, \max\{\delta_1, \delta_3\}\}$.*

Proof. Suppose $\delta_1 = 1$. Then we have that $\delta_3 - i = -i\delta_1 + 1\delta_3$ for $i \in \{0, \ldots, \delta_3 - 2\}$ and $1 = 1\delta_1 + 0\delta_3$. We observe that all this couples are Bézout couples and by Remark 11, we obtain that they are all irreducible. □

Another important fact that should be highlighted is that $\Delta(S)$ does not depend on the generators of S, but on δ_1 and δ_3. So nonsymmetric numerical semigroups with embedding dimension three and with the same δ_1 and δ_3 will have the same Delta sets.

3. Euclid's Algorithm and Delta sets

Let δ_j and δ_k be integers with $1 < \delta_j < \delta_k$ and $\{j, k\} = \{1, 3\}$. We will highlight the fact that $\boldsymbol{x}_i = (x_{ik}, x_{ij})$ is a Bézout couple by explicitly saying that $\boldsymbol{x}_i$ is a Bézout couple for (δ_j, δ_k). In this setting $0 < i \le \delta_k$, $i = x_{ik}\delta_k + x_{ij}\delta_j$, $0 \le |x_{ik}| \le \delta_j$ and $0 \le |x_{ij}| \le \delta_k$.

Denote by $\lfloor x\rfloor$ the largest integer less than x.

Lemma 28. *Let $\boldsymbol{x}_i = (x_{ik}, x_{ij})$ be a Bézout couple for (δ_k, δ_j) with $i \le \delta_j$. Then*

$$\boldsymbol{x}'_i = (\lfloor \delta_k/\delta_j\rfloor x_{ik} + x_{ij}, x_{ik})$$

is a Bézout couple for $(\delta_j, \delta_k \bmod \delta_j)$.

Proof. Notice that, for $i < \delta_j$,

$$(\lfloor \delta_k/\delta_j\rfloor x_{ik} + x_{ij})\, \delta_j + x_{ik}(\delta_k \bmod \delta_j) = x_{ik}\delta_k + x_{ij}\delta_j = i.$$

For $i = \delta_j$, we have $\boldsymbol{x}_{\delta_j} = (0, 1)$ and $\boldsymbol{x}'_{\delta_j} = (1, 0)$. We check that $\boldsymbol{x}'_i$ are indeed Bézout couples for $(\delta_j, \delta_k \bmod \delta_j)$, $i < \delta_j$.

As $0 < x_{ik}\delta_k + x_{ij}\delta_j = i < \delta_j$, dividing by δ_j we obtain

$$0 < (\delta_k/\delta_j)x_{ik} + x_{ij} < 1.$$

Since $0 < x_{ik}\delta_k + x_{ij}\delta_j < \delta_j$ and $\delta_k = \lfloor \delta_k/\delta_j \rfloor \delta_j + \delta_k \bmod \delta_j$, we have $0 < x_{ik}(\lfloor \delta_k/\delta_j \rfloor \delta_j + \delta_k \bmod \delta_j) + x_{ij}\delta_j < \delta_j$. Dividing again by δ_j we obtain

$$0 < (\lfloor \delta_k/\delta_j \rfloor x_{ik} + x_{ij}) + (x_{ik}/\delta_j)(\delta_k \bmod \delta_j) < 1.$$

We distinguish two cases depending on the sign of x_{ik}.

- For $x_{ik} > 0$, as $\boldsymbol{x}_i$ is a Bézout couple, we know that $x_{ik} \le \delta_j$. Observe that $\lfloor \delta_k/\delta_j \rfloor x_{ik} + x_{ij} < (\delta_k/\delta_j)x_{ik} + x_{ij} < 1$. So $\lfloor \delta_k/\delta_j \rfloor x_{ik} + x_{ij} \le 0$, which gives us, using the second set of inequalities, and $x_{ik} \le \delta_j$:
$$0 \le -(\lfloor \delta_k/\delta_j \rfloor x_{ik} + x_{ij}) < (x_{ik}/\delta_j)(\delta_k \bmod \delta_j) \le \delta_k \bmod \delta_j.$$
We have obtained that both coordinates of $\boldsymbol{x}'_i$ satisfy $-(\delta_k \bmod \delta_j) < \lfloor \delta_k/\delta_j \rfloor x_{ik} + x_{ij} \le 0$ and $0 < x_{ik} \le \delta_j$.
- While, if $x_{ik} \le 0$, as $\boldsymbol{x}_i$ is a Bézout couple, we have $x_{ik} > -\delta_j$. Also, $\lfloor \delta_k/\delta_j \rfloor x_{ik} + x_{ij} \ge (\delta_k/\delta_j)x_{ik} + x_{ij} > 0$. So, from the second set of inequalities we have $\lfloor \delta_k/\delta_j \rfloor x_{ik} + x_{ij} < 1 - (x_{ik}/\delta_j)(\delta_k \bmod \delta_j)$, which yields
$$0 < \lfloor \delta_k/\delta_j \rfloor x_{ik} + x_{ij} < 1 + |(x_{ik}/\delta_j)|\delta_k \bmod \delta_j < 1 + \delta_k \bmod \delta_j.$$
Deducing that $0 < \lfloor \delta_k/\delta_j \rfloor x_{ik} + x_{ij} \le \delta_k \bmod \delta_j$, and as $-\delta_j < x_{ik} \le 0$ we obtain again that $\boldsymbol{x}'_i$ is a Bézout couple for $(\delta_j, \delta_k \bmod \delta_j)$. □

Remark 29. From the proof of Lemma 28, it follows that

- if $\boldsymbol{x}_i$ is a λ-Bézout couple, then $\boldsymbol{x}'_i$ is a μ-Bézout couple;
- if $\boldsymbol{x}_i$ is a μ-Bézout couple, then $\boldsymbol{x}'_i$ is a λ-Bézout couple.

The above construction can be reversed.

Lemma 30. *Let i be a natural number $i \le \delta_j$, and we consider $\boldsymbol{x}'_i = (x'_{ij}, x'_{ik})$ a Bézout couple for $(\delta_j, \delta_k \bmod \delta_j)$. Then $\boldsymbol{x}_i = (x'_{ik}, x'_{ij} - x'_{ik}\lfloor \delta_k/\delta_j \rfloor)$ is a Bézout couple for (δ_k, δ_j).*

Proof. It is clear that for $\boldsymbol{x}'_{\delta_j} = (1, 0)$ we obtain $\boldsymbol{x}_{\delta_j} = (0, 1)$. So, we can consider $i < \delta_j$. It is also easy to check that $x'_{ik}\delta_k + (x'_{ij} - x'_{ik}\lfloor \delta_k/\delta_j \rfloor)\delta_j = x'_{ij}\delta_j + x'_{ik}(\delta_k - \lfloor \delta_k/\delta_j \rfloor \delta_j) = x'_{ij}\delta_j + x'_{ik}(\delta_k \bmod \delta_j) = i$.

From this, we have $0 < x'_{ij}\delta_j + x'_{ik}(\delta_k \bmod \delta_j) < \delta_j$. Dividing by δ_j we obtain:

$$0 < x'_{ij} + (x'_{ik}/\delta_j)(\delta_k \bmod \delta_j) < 1.$$

From here, and using that $\delta_k \bmod \delta_j = \delta_k - \lfloor \delta_k/\delta_j \rfloor \delta_j$, we have $0 < x'_{ij} + (x'_{ik}/\delta_j)(\delta_k - \lfloor \delta_k/\delta_j \rfloor \delta_j) < 1$, or equivalently

$$0 < x'_{ij} - \lfloor \delta_k/\delta_j \rfloor x'_{ik} + (x'_{ik}/\delta_j)\delta_k < 1.$$

As in Lemma 28, we distinguish two cases.

- If $x'_{ik} \le 0$, we know that $-\delta_j < x'_{ik}$ and we have $(x'_{ik}/\delta_j)\delta_k \le 0$. Then $x'_{ij} - \lfloor \delta_k/\delta_j \rfloor x'_{ik} > 0$, because both summands are positive. So we have $0 < x'_{ij} - \lfloor \delta_k/\delta_j \rfloor x'_{ik} < 1 - (x'_{ik}/\delta_j)\delta_k < 1 + \delta_k$, since $-x'_{ik} = |x'_{ik}| < \delta_j$. Hence $x'_{ij} - \lfloor \delta_k/\delta_j \rfloor x'_{ik} \le \delta_k$. Obtaining that $\boldsymbol{x}_i$ is a Bézout couple.
- If $x'_{ik} > 0$, we know that $x'_{ik} \le \delta_j$ and, from the first equation, we have that $x'_{ij} < 1$, or equivalently $x'_{ij} \le 0$. So we can write, $0 \le -x'_{ij} < (x'_{ik}/\delta_j)(\delta_k \bmod \delta_j) = (x'_{ik}/\delta_j)(\delta_k - \lfloor \delta_k/\delta_j \rfloor \delta_j) = (x'_{ik}/\delta_j)\delta_k - \lfloor \delta_k/\delta_j \rfloor x'_{ik}$. Adding $\lfloor \delta_k/\delta_j \rfloor x'_{ik}$, we obtain $0 < \lfloor \delta_k/\delta_j \rfloor x'_{ik} \le \lfloor \delta_k/\delta_j \rfloor x'_{ik} - x'_{ij} < (x'_{ik}/\delta_j)\delta_k < \delta_k$, as $x'_{ik} < \delta_j$. So, $-\delta_k < x'_{ij} - \lfloor \delta_k/\delta_j \rfloor x'_{ik} \le 0$ and as $0 < x'_{ik} \le \delta_j$. This proves that $\boldsymbol{x}_i$ is a Bézout couple. □

Definition 31. *Set $\mathcal{B}(\delta_k, \delta_j) = \{\boldsymbol{x}_i \mid \boldsymbol{x}_i$ is a Bézout couple for $(\delta_k, \delta_j)\}$.*

Proposition 32. *The map $f\colon \mathcal{B}(\delta_j, \delta_k \bmod \delta_j) \to \mathcal{B}(\delta_k, \delta_j)$, $\boldsymbol{x}'_i \mapsto \boldsymbol{x}_i$, coming from Lemma 30, is additive and injective.*

Proof. Follows easily from the definition of $\boldsymbol{x}_i$ in Lemma 30 and Lemma 28. $\square$

Remark 33. Note that the elements in $\mathrm{Im}(f)$ correspond with Bézout couples for numbers smaller than or equal than δ_j.

Now, we want to prove that the irreducible Bézout couples in $\mathcal{B}(\delta_k,\delta_j)\setminus\mathrm{Im}(f)$ are those with $x_{ik}=1$. First of all, we compute these Bézout couples in the next proposition.

Proposition 34. *Let $\boldsymbol{x}_i=(1,x_{ij})\in\mathcal{B}(\delta_k,\delta_j)$. Then $\boldsymbol{x}_i\notin\mathrm{Im}(f)$ if and only if $-\lfloor\delta_k/\delta_j\rfloor<x_{ij}\le 0$.*

Proof. Take $\boldsymbol{x}_i=(1,x_{ij})\in\mathcal{B}(\delta_k,\delta_j)$. Hence $\delta_k+x_{ij}\delta_j=i\le\delta_k$, and $x_{ij}\le 0$.

If $x_{ij}>-\lfloor\delta_k/\delta_j\rfloor$, then $0\ge x_{ij}\ge-\lfloor\delta_k/\delta_j\rfloor+1$. Hence $i\ge\delta_k-\lfloor\delta_k/\delta_j\rfloor\delta_j+\delta_j=(\delta_k \bmod \delta_j)+\delta_j>\delta_j$. Therefore $\boldsymbol{x}_i\notin\mathrm{Im}(f)$.

It is clear that $\delta_k+x_{ij}\delta_j$ with $x_{ij}\le-\lfloor\delta_k/\delta_j\rfloor$ are elements smaller than δ_j and by Lemma 28 the corresponding $\boldsymbol{x}_i$ is in $\mathrm{Im}(f)$. $\square$

Remark 35. Observe that the case $x_{ik}=1$ and $x_{ij}=-\lfloor\delta_k/\delta_j\rfloor$, yields an irreducible couple with $x_{ik}\delta_k+x_{ij}\delta_j=\delta_k-\lfloor\delta_k/\delta_j\rfloor\delta_j=\delta_k \bmod \delta_j<\delta_j$.

Proposition 36. *Irreducible elements in $\mathcal{B}(\delta_k,\delta_j)\setminus\mathrm{Im}(f)$ are those Bézout couples with $x_{ik}=1$ and $-\lfloor\delta_k/\delta_j\rfloor<|x_{ij}|\le 0$.*

Proof. From Remark 11 we know that Bézout couples of the form $(1,x_{ij})$ are irreducible. We need to prove that all elements bigger than δ_j different from these have associated reducible Bézout couples.

For this, suppose that $\delta_k>i>\delta_j$. Now, we can find $q\le\lfloor\delta_k/\delta_j\rfloor$ such that $\delta_k-q\delta_j>i>\delta_k-(q+1)\delta_j>0$; in particular, $i=\delta_k-(q+1)\delta_j+r$ with r a positive integer such that $r<\delta_j$. From Remark 11, Proposition 34 and Remark 35, we have that $\boldsymbol{x}_{i'}=(1,-(q+1))$ is an irreducible Bézout couple for some $i'<i$.

If $x_{ik}>1$, consider the Bézout couple $\boldsymbol{x}_r=(x_{rk},x_{rj})$ associated to r with $0<x_{rk}\le\delta_j$ (and $-\delta_k<x_{rj}\le 0$; Lemma 9). If $x_{rk}=\delta_j$, then $r=\delta_j\delta_k+x_{rj}\delta_j=\delta_j(\delta_k+x_{rj})\ge\delta_j$, contradicting that $r<\delta_j$. Hence $x_{rk}<\delta_j$ and $\boldsymbol{x}_i=(1,-(q+1))+\boldsymbol{x}_r$, obtaining that $\boldsymbol{x}_i$ is not irreducible.

If $x_{ik}<0$, write $i=\delta_j+r$. We consider now the Bézout couple $\boldsymbol{x}_r$ associated to r with $-\delta_j<x_{rk}\le 0$ and $0<x_{rj}\le\delta_k$. Then $\boldsymbol{x}_i=(0,1)+(x_{rk},x_{rj})$, if $x_{rj}+1\le\delta_k$, or equivalently, $x_{rj}\ne\delta_k$. If $x_{rj}=\delta_k$, then $r=x_{rk}\delta_k+\delta_k\delta_j=\delta_k(x_{rk}+\delta_j)\ge\delta_k$, a contradiction.

If $x_{ik}=0$ then $i=t\delta_j$. Then, if $t=1$ we have $i=\delta_j\in\mathrm{Im}(f)$ and for $t>1$ we deduce $(0,t)=(0,t-1)+(0,1)$ obtaining again that (x_{ik},x_{ij}) is a reducible Bezout couple. $\square$

Proposition 37. *Irreducible Bézout couples of $\mathcal{B}(\delta_k,\delta_j)\cap\mathrm{Im}(f)$ are only those that come from irreducible Bézout couples in $\mathcal{B}(\delta_j,\delta_k \bmod \delta_j)$.*

Proof. Irreducible elements in $\mathrm{Im}(f)$ are those elements in $\mathrm{Im}(f)$ that can not be written as sum of two elements in $\mathcal{B}(\delta_k,\delta_j)$. So additivity of f ensures that the pre-images of these irreducible elements can not be expressed as sum of elements in $\mathcal{B}(\delta_j,\delta_k \bmod \delta_j)$.

If we have $\boldsymbol{x}'_i$ an irreducible element in $\mathcal{B}(\delta_j,\delta_k \bmod \delta_j)$, we know that $i<\delta_j$ so we can not to write $\boldsymbol{x}_i$ as sum of elements out of $\mathrm{Im}(f)$, because elements out of $\mathrm{Im}(f)$ correspond with numbers bigger than δ_j. $\square$

Observe that according to the last two results and Theorem 21 the elements in $\Delta(S)$ can be obtained in the following way.

- First compute the couples $(1,-t)$ with $t\in\{0,\dots,\lfloor\delta_3/\delta_1\rfloor\}$. These correspond to the values $\delta_3-t\delta_1$ in $\Delta(S)$.
- Then proceed in the same way with $(\delta_3,\delta_3 \bmod \delta_1)$, until we reach $\gcd(\delta_1,\delta_3)$.

Observe that if $\delta_1<\delta_3$, then $\lfloor\delta_1/\delta_3\rfloor=0$, and we go directly to the second step, swapping δ_1 with δ_3.

Thus the possible values in $\Delta(S)$ are those arising in the calculation of $\gcd(\delta_1, \delta_3)$ in the naive way. That is $\gcd(a, b) = \gcd(a, b - a)$ while $b > a$, and then we swap positions and start anew, until we reach 0 in the second argument. The output is just the set of a's and b's appearing in the process removing 0.

Example 38. Let $S = \langle 1407, 26962, 35413\rangle$. Its minimal presentation is:

$$\{((411, 0, 0), (0, 7, 11)); ((0, 91, 0), (284, 0, 58)); ((0, 0, 69), (127, 84, 0))\},$$

so $\delta_1 = 393$, $\delta_2 = 251$, $\delta_3 = 142$. We start the Euclid's algorithm with $\delta_k = 393$ and $\delta_j = 142$ to obtain:

$$\begin{array}{cccccc}
\delta_k = 393 & \delta_j = 142 & & & & \\
& \overset{(1,0)}{393} & \overset{(1,-1)}{251} & \overset{(1,-2)}{109} & = \delta_k \bmod \delta_j & \\
\delta_k = 142 & \delta_j = 109 & & & & \\
& \overset{(0,1)}{142} & \overset{(-1,3)}{33} & = \delta_k \bmod \delta_j & & \\
\delta_k = 109 & \delta_j = 33 & & & & \\
& \overset{(1,-2)}{109} & \overset{(2,-5)}{76} & \overset{(3,-8)}{43} & \overset{(4,-11)}{10} & = \delta_k \bmod \delta_j \\
\delta_k = 33 & \delta_j = 10 & & & & \\
& \overset{(-1,3)}{33} & \overset{(-5,14)}{23} & \overset{(-9,25)}{13} & \overset{(-13,36)}{3} & = \delta_k \bmod \delta_j \\
\delta_k = 10 & \delta_j = 3 & & & & \\
& \overset{(4,-11)}{10} & \overset{(17,-47)}{7} & \overset{(30,-83)}{4} & \overset{(43,-119)}{1} & = \delta_k \bmod \delta_j \\
\delta_k = 3 & \delta_j = 1 & & & & \\
& \overset{(-13,36)}{3} & \overset{(-56,155)}{2} & \overset{(-99,274)}{1} & \overset{(-142,393)}{0} & = \delta_k \bmod \delta_j
\end{array}$$

The coordinates shown over the integers are coordinates with respect to $\delta_k = 393$ and $\delta_j = 142$. So, we have that the μ-Bézout couples are:

$$\{(1, 0), (1, -1), (1, -2), (2, -5), (3, -8), (4, -11), (17, -47), (30, -83), (43, -119)\}$$

and the λ-Bézout couples are:

$$\{(0, 1), (-1, 3), (-5, 14), (-9, 25), (-13, 36), (-56, 155), (-99, 274)\}.$$

Observe that the couples $(43, -119)$ and $(-99, 274)$ correspond to $i = 1$, and both are irreducible Bézout couples, but we only need to compute them once, so we can "forget" $(43, -119)$ when we are looking for the Delta set.

Thus the Delta set for S is:

$$\Delta(S) = \{1, 2, 3, 4, 7, 10, 13, 23, 33, 43, 76, 109, 142, 251, 393\}.$$

In practice, when we are only interested in the Delta set, we do not need to keep track of the Bézout couples, just the integers appearing in the greatest common divisor computation.

```
gap> deltasetnsembdim3(1407, 26962, 35413);time;
[ 1, 2, 3, 4, 7, 10, 13, 23, 33, 43, 76, 109, 142, 251, 393 ]
1
```

The time is in milliseconds, that is, it takes 1 millisecond to compute $\Delta(S)$. The current procedure `DeltaSetOfNumericalSemigroup` in `numericalsgps` executed with this example was stopped after one day without an output. The implementation of `DeltaSetOfNumericalSemigroup` is based on a dynamical procedure presented in [3] and was kindly programmed by Chris O'Neil (see the

contributions section in the manual of `numericalsgps`). Of course it was meant for arbitrary numerical semigroups, and not just nonsymmetric numerical semigroups with embedding dimension three. The idea in [3] is to compute all Delta sets of elements up to when this calculation becomes periodical. In our example the bound for periodicity is just too big; this is why it was not able to give an answer after one day of computation.

```
gap> DeltaSetPeriodicityBoundForNumericalSemigroup(
> NumericalSemigroup(1407, 26962, 35413));
916982754
```

Next we show another examples of execution with their timings (milliseconds).

```
gap> s:=NumericalSemigroup(101,301,510);;
gap> DeltaSetOfNumericalSemigroup(s);time;
[ 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18 ]
10271
gap> deltasetnsembdim3(101,301,510);time;
[ 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18 ]
1
gap> s:=NumericalSemigroup(151,301,510);;
gap> DeltaSetOfNumericalSemigroup(s);time;
[ 1, 2, 3, 5, 7, 12, 17, 22 ]
4976
gap> deltasetnsembdim3(151,301,510);time;
[ 1, 2, 3, 5, 7, 12, 17, 22 ]
1
gap> deltasetnsembdim3(8,41,79);time;
[ 1, 2, 3, 4, 5, 6, 7, 13 ]
1
```

In [1, Section 2] we present a procedure to compute the primitive elements of $\ker\varphi$, or equivalently, a Graver basis of M, that is, the set of minimal nonzero elements of M with respect to $\sqsubseteq$, defined as $(x_1, x_2, x_3) \sqsubseteq (y_1, y_2, y_3)$ if $x_i y_i \geq 0$ and $|x_i| \leq |y_i|$ for $i \in \{1, 2, 3\}$. It is easy to prove that $\mathbf{v}_1$, $\mathbf{v}_2$ and $\mathbf{v}_3$ are elements in this Graver basis. So for instance, in order to compute $\mathbf{v}_1$, we look for the elements in the Graver basis of M of the form (a, b, c) with $a \neq 0$ and $bc \geq 0$, and choose the element with least $|a|$ (which corresponds with c_1). In this way we can compute a minimal presentation for S, and consequently δ_1 and δ_3. The algorithm presented in [1] has the same complexity as Euclid's greatest common divisor algorithm. The timings presented in the above examples for `deltasetembdim3` include the calculation of a minimal presentation.

Checking whether or not S is nonsymmetric can be done easily by using [15, Theorem 10.6], which relies also in greatest common divisor calculations. The semigroup S is symmetric if and only if it is of the form $\langle am_1, am_2, bm_1 + cm_2\rangle$ with

- m_1 and m_2 coprime integers greater than one,
- a, b and c nonnegative integers with $a \geq 2$, $b + c \geq 2$ and $\gcd(a, bm_1 + cm_2) = 1$.

So if we want to check whether or not $S = \langle n_1, n_2, n_3\rangle$ is symmetric, we take all possible partitions $\{\{n_i, n_j\}, \{n_k\}\}$ with $\{i, j, k\} = \{1, 2, 3\}$. Then for each partition we compute $a = \gcd\{n_i, n_j\}$, and if it is greater than one, we check if $n_k \in \langle n_i/a, n_j/a\rangle \setminus \{n_i/a, n_j/a\}$. If so, the semigroup is symmetric. If it is not the case for any partition, then S is not symmetric.

Example 39. Let $S = \langle 4, 6, 9\rangle$. Then $\gcd(4, 6) = 2$ and $9 \in \langle 2, 3\rangle \setminus \{2, 3\}$. Whence S is symmetric.

For $S = \langle 3, 5, 7\rangle$, every two generators are coprime (and so $a = 1$), whence S is not symmetric. If we compute the Graver basis for $M \equiv 3x + 5y + 7z = 0$ using [1], we obtain

$$G = \{(0, -7, 5), (1, -2, 1), (1, 5, -4), (2, 3, -3), (3, 1, -2), (4, -1, -1), (5, -3, 0), (7, 0, -3)\}$$

(we remove $-\mathbf{v}$ if $\mathbf{v}$ is already in the basis). When looking for $\mathbf{v}_1$ we need to search for the elements $(a, b, c) \in G$ with $a \neq 0$ and $bc \geq 0$: $\{(4, -1, -1), (5, -3, 0), (7, 0, -3)\}$. Then choose the element with minimal $|a|$. In this case, $\mathbf{v}_1 = (4, -1, -1)$, which yields $((4, 0, 0), (0, 1, 1)) \in \ker \phi$. We proceed in the same way with $\mathbf{v}_2$ and $\mathbf{v}_3$. It follows that a minimal presentation is

$$\{((4, 0, 0), (0, 1, 1)), ((0, 2, 0), (1, 0, 1)), ((0, 0, 2), (3, 1, 0))\}.$$

Hence $\delta_1 = 2 = \delta_3$ and $\delta_2 = 0$. In this case $\Delta(S) = \{2\}$.

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Departamento de Álgebra and CITIC-UGR, Universidad de Granada, 18071 Granada, España
E-mail address: `pedro@ugr.es`

Departamento de Matemáticas, Universidad de Almería, 04120 Almería, España
E-mail address: `dllena@ual.es`

Dipartimento di Matematica e Informatica, Università di Catania, Viale Andrea Doria 6, and Scuola Superiore di Catania, Università di Catania, Via Valdisavoia 9, 95125 Catania, Italy.
E-mail address: `alessio.moscariello@studium.unict.it`